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CAN THE THEORY OF LINEAR VISCOELASTICITY BE DERIVED FROM THE CURRENT THERMODYNAMIC THEORY OF SIMPLE MATERIALS WITH FADING MEMORY? AN OBJECTION

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Abstract—This paper deals with some consequences of the thermodynamic approach[2, 3] to the theory of simple materials with fading memory. It is proved that this approach implies that the time derivative of the stress–relaxation modulus appearing in the isothermal first-order approximation of the theory must vanish identically. This invalidates the claim that the linear theory of viscoelasticity can be considered as the isothermal first-order approximation of the theory presented in[2, 3].

1. INTRODUCTION

The theory of simple materials with fading memory has been reviewed in an extensive monograph by Dill[4]. To this lucid and comprehensive work the reader is referred for a complete discussion on the subject, for quotations of original papers and for a fuller explanation of the notation adopted here. It was shown by Coleman and Noll in a paper dating back to 1961[1] that, when a purely mechanical approach is followed, the finite linear theory of viscoelasticity and thus, in particular, the classical infinitesimal theory of viscoelasticity, can be considered as an appropriate first-order approximation of the theory of simple materials with fading memory. Subsequently, in two papers dated 1964, Coleman[2, 3] gave a thermodynamic structure to the mechanical theory of simple materials with fading memory. Among other things, Coleman showed that the mechanical theory of linear viscoelasticity can be viewed as the isothermal first-order approximation of the theory he proposed. This result was derived in[3] by an analysis which closely parallels that pursued by Coleman and Noll[1] for the purely mechanical theory (see the remark in the footnote of p. 247 of Ref. [3]). The thermodynamic theory, however, allowed Coleman to go further and find restrictions on the form of the stress–relaxation modulus appearing in the first-order approximation he established. Yet, the ultimate consequences of this thermodynamical approach appear to have not been considered. It will be shown in this paper that, as a consequence of Coleman's thermodynamic approach to simple materials with fading memory, the time-derivative of the above mentioned stress–relaxation modulus must vanish identically. This invalidates the claim that the linear theory of viscoelasticity can be considered as an approximation of the thermodynamic theory proposed in[3, 4]. The same criticism, however, does not apply to the purely mechanical theory presented in[1].

In the original paper[3] Coleman referred to infinitesimal linear viscoelasticity. His analysis, however, holds true also for the more general case of finite linear viscoelasticity (see [4], pp. 363–367]). For this reason the case of finite linear viscoelasticity will be treated in what follows. The results to be proved, however apply also to the particular case of infinitesimal linear viscoelasticity, since they are not dependent on the magnitude of the deformations involved. Of course, the strain history of the material is intended to be sufficiently near to a constant deformation history so that a first-order approximation of the general constitutive equations can be taken.

By *magnitude* of a second order tensor **T** we shall understand the quantity $|\mathbf{T}|$ defined by

$$|\mathbf{T}| = \mathrm{tr}(\mathbf{T}\mathbf{T}^T)^{1/2}. \tag{1.1}$$

The distance between two deformation histories will be measured in the functional space of all the histories endowed with the h-*norm* introduced in [2, 3]. In particular, the h-norm in the space of all the deformation histories $\mathbf{E}_r^t(s)$ defined in the interval $0 < s < \infty$ (*past histories*) is intended to be given by

$$\|\mathbf{E}_r^t(s)\|_h^2 = \int_0^\infty |\mathbf{E}_r^t(s)|^2 h(s)^2 \, ds. \quad (1.2)$$

The function $h(s)$ appearing in this relation is assumed to be positive, continuous and monotone decreasing in the interval $0 \leq s < \infty$. Moreover, $h(s)$ is supposed to be such that

$$\lim_{s\to\infty} s^{1/2+\delta} h(s) = 0, \quad \text{monotonically for large } s, \quad (1.3)$$

for some $\delta > 0$. From these assumptions it follows that $h(s)$ is bounded from above and from below in the entire interval $0 \leq s < \infty$. Following Coleman[3], it will be assumed that the constitutive functionals defining a simple material with fading memory are continuously Fréchet differentiable and continuously derivable in the ordinary sense up to the order $n \geq 2$ in their common domain of definition.

2. RESULTS FROM THE THERMODYNAMIC THEORY OF SIMPLE MATERIALS WITH FADING MEMORY

Let S denote the *rotated stress tensor* defined by $\mathbf{S} = \mathbf{R}\mathbf{t}\mathbf{R}^T$, the tensors $\mathbf{R}$ and t being respectively the roatation tensor and the Cauchy stress tensor. If F denotes the deformation gradient and U is the right stretch tensor, then $\mathbf{R}$ is defined by the relation $\mathbf{F} = \mathbf{R}\mathbf{U}$. The strain measure $\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1})$ will be adopted in what follows. For a given material point the *past history* of E up to time t will be henceforth denoted by $\mathbf{E}_r^t = \mathbf{E}_r^t(s) = \mathbf{E}(t-s)$, $0 < s < \infty$, while the past history of the absolute temperature θ up to time t will be denoted by $\vartheta_r^t = \vartheta_r^t(s) = \theta(t-s)$, $0 < s < \infty$. The values of strain and temperature at time t will be simply denoted by E and θ.

In the thermodynamic theory of simple materials with fading memory, the constitutive equations for the specific free energy ψ and for the stress tensors S are found to be given by the following functionals (see, e.g. [4], Section 4.10.1)

$$\psi = \bar{\psi}(\mathbf{E}_r^t, \theta_r^t; \mathbf{E}, \theta), \quad (2.1)$$

$$\mathbf{S} = \rho \mathbf{U} D_{\mathbf{E}} \bar{\psi}(\mathbf{E}_r^t, \theta_r^t; \mathbf{E}, \theta). \quad (2.2)$$

Here ρ denotes mass density while the symbol $D_{\mathbf{E}}$ stands for the ordinary partial derivative with respect to E. It is found, moreover, that

$$\sigma \stackrel{\text{def}}{=} -\delta_\theta \bar{\psi}(\mathbf{E}_r^t, \theta_r^t; \mathbf{E}, \theta | \dot{\theta}_r^t) - \delta_{\mathbf{E}} \bar{\psi}(\mathbf{E}_r^t, \theta_r^t; \mathbf{E}, \theta | \dot{\mathbf{E}}_r^t) \geq 0 \quad (2.3)$$

for every thermodynamic process. The quantity σ defined by (2.3) is called *internal dissipation.* The functionals $\delta_\theta \bar{\psi}(\mathbf{E}_r^t, \theta_r^t; \mathbf{E}, \theta | \cdot)$ and $\delta_{\mathbf{E}} \bar{\psi}(\mathbf{E}_r^t, \theta_r^t; \mathbf{E}, \theta | \cdot)$ are the Fréchet derivatives of the functional $\bar{\psi}$ with respect to the function arguments θ_r^t and $\mathbf{E}_r^t$, respectively. The present analysis will be confined to processes occurring at constant θ for all times before t. For these processes the temperature history reduces to the constant history $\theta^+ = \theta(s) \equiv \theta = \text{const}$ for all $0 \leq s < \infty$ and, accordingly, the dependence on the constant parameter θ will be understood in most of the formulae which follow.

The constitutive equation for S in a linear viscoelastic material can be expressed by

$$\mathbf{S}(t) = \mathbf{S}_\infty(\mathbf{E}) + \int_0^\infty \dot{\mathbf{K}}(\mathbf{E}, s) \mathbf{E}_d^t(s) \, ds, \quad (2.4)$$

where $S_\infty(E)$ is the equilibrium value of S, while the fourth-order tensor $K(E, s)$ is the so-called *stress-relaxation modulus.* It can be shown (see, e.g. [4], Section 4.7) that eqn (2.4) may be regarded as the isothermal first-order approximation of the general constitutive eqn (2.2) for small *difference histories* $E_d^t(s) \overset{def}{=} E_r^t(s) - E$, $0 < s < \infty$. In particular, the integral in (2.4) can be shown to be such that ([4], eqn 4.11.41)

$$-\frac{1}{\rho}\operatorname{tr}\left[U^{-1}\int_0^\infty \dot{K}(E, s)\, A(s)\, ds\, U^{-1}M\right] = \frac{d}{d\alpha}\frac{d}{d\beta}\bar{\psi}(E^+ + \alpha A(s) + \beta M^+; E)\Big|_{\substack{\alpha=0\\ \beta=0}} \tag{2.5}$$

for any symmetric tensor M and for any past history A(s). In eqn (2.5) the quantities E^+ and M^+ denote the constant strain histories $E^+ = E(s) \equiv E$ and $M^+ = M(s) \equiv M$, $0 \le s < \infty$.

Let $\bar{\sigma}$ denote the internal dissipation relevant to processes in which the temperature history is the constant history θ^+. Since the functionals $\delta_\theta\bar{\psi}$ and $\delta_E\bar{\psi}$ appearing in (2.3) are linear in their last arguments, for a process occurring at constant temperature the inequality (2.3) can be written in the form

$$\bar{\sigma}(t) = -\delta_E\bar{\psi}(E_r^t, \theta^+; E, \theta|\dot{E}_r^t) = -\Phi(E_r^t; E)\dot{E}_r^t \ge 0, \tag{2.6}$$

where we have denoted by $\Phi(E_r^t; E)$ a linear operator† (the Fréchet derivative of $\bar{\psi}$ with respect to E_r^t) representing a mapping from the Hilbert space of all the functions E_r^t into the set of the real numbers. Henceforth, the null element in a Banach space of linear operators will be denoted by ϕ.

3. PROOF THAT $\dot{K}(E, s) \equiv 0$

From the above premises, it will be proved in this section that the considered thermodynamic theory of simple materials with fading memory leads to the conclusion that the time-derivative of the stress–relaxation modulus K(E, s) must vanish for every value of E and s. This implies that the isothermal first-order approximation of (2.2) for small difference histories is, in fact, $S = S_\infty(E)$. This approximation, therefore, cannot be regarded as the constitutive equation of a linear viscoelastic material.

LEMMA 1. *For any constant strain history* $E^+ = E$, *for any symmetric tensor* H *and for any arbitrary small* $\epsilon > 0$, *it is possible to determine a scalar* $k > 0$ *in such a way that the past strain history defined by*

$$H_r^t = H_r^t(s) = kHs^{1/2+\delta} + E^+ \tag{3.1}$$

is such that

$$\|H_r^t - E^+\|_h \le \epsilon. \tag{3.2}$$

Proof. From definition (1.2) and from eqn (3.1) it follows that

$$\|H_r^t - E^+\|_h^2 = \int_0^\infty [s^{1/2+\delta}h(s)]^2\, ds \tag{3.3}$$

The integral in this equation is positive and finite because $h(s)$ is a continuous positive monotone decreasing function in $[0, \infty)$, because $\delta > 0$ and because relation (1.3) holds. Therefore,

$$\int_0^\infty [s^{1/2+\delta}h(s)]^2\, ds = a^2 \tag{3.4}$$

†The dependence on the constant parameter θ being understood.

where $0 < a^2 < \infty$. It turns out that relation (3.2) is satisfied when

$$k \leq \epsilon^{1/2}(a|\mathbf{H}|)^{-1}; \tag{3.5}$$

which proves the lemma.

LEMMA 2. *From inequality* (2.6) *and from the assumed hypotheses of differentiability of* $\bar{\psi}$, *it follows that*

$$\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}) \equiv \boldsymbol{\phi} \tag{3.6}$$

for all constant strain histories $\mathbf{E}^+ = \mathbf{E}$.

Proof. By applying eqn $(2.6)_2$ to the strain history (3.1) we get

$$\bar{\sigma}(t) = +\left(\frac{1}{2} + \delta\right) k\mathbf{\Phi}(k\mathbf{H}s^{1/2+\delta} + \mathbf{E}^+; \mathbf{E})\mathbf{H}s^{\delta-1/2}, \tag{3.7}$$

since

$$\mathbf{H}_r^t(0) = \mathbf{E} \quad \text{and} \quad \dot{\mathbf{H}}_r^t(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{H}_r^t(s) = -\left(\frac{1}{2} + \delta\right) ks^{\delta-1/2}\mathbf{H}. \tag{3.8}$$

The functional $\bar{\psi}$ is assumed to be at least twice Fréchet differentiable. It follows that the functional $\mathbf{\Phi}(\mathbf{E}_r^t; \mathbf{E})$ is Fréchet differentiable and, thus, the relation

$$\begin{aligned}\mathbf{\Phi}(k\mathbf{H}s^{1/2+\delta} + \mathbf{E}^+; \mathbf{E}) &= \mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}) + \delta_{\mathbf{E}}\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}|k\mathbf{H}s^{1/2+\delta}) + o(\|k\mathbf{H}s^{1/2+\delta}\|_h)\\ &= \mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}) + \delta_{\mathbf{E}}\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}|k\mathbf{H}s^{1/2+\delta}) + o(\|\mathbf{H}_r^t(s) - \mathbf{E}^+\|_h)\end{aligned} \tag{3.9}$$

holds. Here, of course, the symbol $o(\|\mathbf{A}\|_h)$ denotes an operator with the property that the expression $(\|\mathbf{A}\|_h)^{-1}o(\|\mathbf{A}\|_h)$ tends to $\boldsymbol{\phi}$ as $\|\mathbf{A}\|_h \to 0$. If the history $\mathbf{H}_r^t$ is sufficiently near to $\mathbf{E}^+$, from (3.9), from (3.7) and from inequality (2.6) we get that

$$\bar{\sigma}(s) = -\left(\frac{1}{2} + \delta\right) k[\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}) + \delta_{\mathbf{E}}\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}|k\mathbf{H}s^{1/2+\delta})]\mathbf{H}s^{\delta-1/2} \geq 0. \tag{3.10}$$

The l.h.s. of this inequality contains both a linear term in $\mathbf{H}$ and a quadratic one because $\delta_{\mathbf{E}}\mathbf{\Phi}$ is linear in $\mathbf{H}$. Since (3.10) has to be satisfied for all symmetric $\mathbf{H}$, the linear term in $\mathbf{H}$ must vanish; hence (3.6) follows (see also eqn $(4.10.18)_2$ of [4]).

THEOREM. *If the stress-relaxation modulus* $\mathbf{K} = \mathbf{K}(\mathbf{E}, s)$, *regarded as a function of* s, *is both integrable in every finite sub-interval of* $[0, \infty)$ *and bounded in the entire interval* $[0, \infty)$, *then*

$$\dot{\mathbf{K}}(\mathbf{E}, s) \equiv \mathbf{0}, \tag{3.11}$$

where $\mathbf{0}$ *is the null fourth-order tensor.*

Proof. Since eqn (2.5) is valid for any constant symmetric tensor $\mathbf{M}$ and for any past history $\mathbf{A}(s)$, it must be valid, in particular, for

$$\mathbf{A}(s) \equiv \mathbf{A}_r^t(s) = \mathbf{M}s. \tag{3.12}$$

With this choice for $\mathbf{A}(s)$, eqn (2.5) becomes

$$-\frac{1}{\rho}\mathrm{tr}\left[\mathbf{U}^{-1}\int_0^\infty \dot{\mathbf{K}}(\mathbf{E}, s)\,\mathbf{M}s\,\mathrm{d}s\,\mathbf{U}^{-1}\mathbf{M}\right] = \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\beta}\bar{\psi}(\mathbf{E}^+ + \alpha\mathbf{A}_r^t(s) + \beta\mathbf{M}^+; \mathbf{E})\Big|_{\substack{\alpha=0\\ \beta=0}}. \tag{3.13}$$

Since

$$\dot{\mathbf{A}}_r^t = -\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{A}_r^t(s) = -\mathbf{M} \tag{3.14}$$

and since

$$\overline{\mathbf{E}^+ + \alpha \mathbf{A}_r^t}^{\boldsymbol{\cdot}} = \alpha \dot{\mathbf{A}}_r^t \quad \text{and} \quad \mathbf{E}^+ + \alpha \mathbf{A}_r^t(0) = \mathbf{E}, \tag{3.15}$$

the r.h.s. of (3.13) can be put in the form:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\beta}\bar{\psi}(\mathbf{E}^+ + \alpha \mathbf{A}_r^t + \beta \mathbf{M}^+; \mathbf{E})\Big|_{\substack{\alpha=0\\ \beta=0}} &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\beta}\bar{\psi}(\mathbf{E}^+ + \alpha \mathbf{A}_r^t - \beta \dot{\mathbf{A}}_r^t; \mathbf{E})\Big|_{\substack{\alpha=0\\ \beta=0}} \\ &= -\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{1}{\alpha}\delta_{(\mathbf{E}^+ + \alpha \mathbf{A}_r^t)}\bar{\psi}(\mathbf{E}^+ + \alpha \mathbf{A}_r^t; \mathbf{E}|\alpha \dot{\mathbf{A}}_r^t)\right]_{\alpha=0} \\ &= -\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{1}{\alpha}\delta_{(\mathbf{E}^+ + \alpha \mathbf{A}_r^t)}\bar{\psi}(\mathbf{E}^+ + \alpha \mathbf{A}_r^t; \mathbf{E}|\overline{\mathbf{E}^+ + \alpha \mathbf{A}_r^t}^{\boldsymbol{\cdot}})\right]_{\alpha=0}. \end{aligned} \tag{3.16}$$

In view of eqn $(2.6)_1$, we can set

$$-\delta_{(\mathbf{E}^+ + \alpha \mathbf{A}_r^t)}\bar{\psi}(\mathbf{E}^+ + \alpha \mathbf{A}_r^t; \mathbf{E}|\overline{\mathbf{E}^+ + \alpha \mathbf{A}_r^t}^{\boldsymbol{\cdot}}) = \bar{\sigma}_A(t), \tag{3.17}$$

where $\bar{\sigma}_A(t)$ is the internal dissipation relevant to the past history $\mathbf{E}^+ + \alpha \mathbf{A}_r^t$. From (3.16), (3.17) and (3.13), therefore, it follows that

$$-\frac{1}{\rho}\operatorname{tr}\left[\mathbf{U}^{-1}\int_0^\infty \dot{\mathbf{K}}(\mathbf{E}, s)\mathbf{M}s\,\mathrm{d}s\mathbf{U}^{-1}\mathbf{M}\right] = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{1}{\alpha}\bar{\sigma}_A(t)\right]_{\alpha=0}. \tag{3.18}$$

Let, on the other hand, relation (2.6) be applied to the history

$$\mathbf{E}^t(s) = \mathbf{E}^+ + \alpha \mathbf{B}^t(s), \tag{3.19}$$

where $\mathbf{E}^+$ is the constant history with constant value $\mathbf{E}$, while α is a positive scalar and $\mathbf{B}^t(s)$ is any strain history such that $\mathbf{B}^t(0) = \mathbf{0}$. It will henceforth be assumed that both $\mathbf{B}_r^t$ and $\dot{\mathbf{B}}_r^t$ are bounded in the interval $0 < s < \infty$. Since

$$\dot{\mathbf{E}}_r^t = \alpha \dot{\mathbf{B}}_r^t \quad \text{and} \quad \mathbf{E}^t(0) = \mathbf{E}, \tag{3.20}$$

from (2.6) and (3.19) we get

$$\begin{aligned}\bar{\sigma}(t) = -\mathbf{\Phi}(\mathbf{E}^+ + \alpha \mathbf{B}_r^t; \mathbf{E}) \cdot \alpha \dot{\mathbf{B}}_r^t &= -[\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}) \cdot \alpha \dot{\mathbf{B}}_r^t + \delta_{\mathbf{E}}\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}|\alpha \mathbf{B}_r^t) \cdot \alpha \dot{\mathbf{B}}_r^t + o(\|\alpha \mathbf{B}_r^t\|_h) \cdot \alpha \dot{\mathbf{B}}_r^t] \\ &\geq 0. \end{aligned} \tag{3.21}$$

In view of relation (3.6) and of the fact that $\alpha > 0$, from (3.21) we get

$$\frac{\bar{\sigma}(t)}{\alpha} = -\delta_{\mathbf{E}}\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}|\alpha \mathbf{B}_r^t) \cdot \dot{\mathbf{B}}_r^t - o(\|\alpha \mathbf{B}_r^t\|_h) \cdot \dot{\mathbf{B}}_r^t \geq 0. \tag{3.22}$$

Since the quantity $\delta_{\mathbf{E}}\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}|\alpha \mathbf{B}_r^t)$ is linear in its last argument, it is convenient to introduce the operator $\delta\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E})$ defined by

$$\delta\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}) \cdot \mathbf{A} \stackrel{def}{=} \delta_{\mathbf{E}}\mathbf{\Phi}(\mathbf{E}^+; \mathbf{E}|\mathbf{A}), \tag{3.23}$$

that denotes the Frechét derivative of $\mathbf{\Phi}(\mathbf{E}_r^t; \mathbf{E})$ with respect to $\mathbf{E}_r^t$ calculated for $\mathbf{E}_r^t \equiv \mathbf{E}^+$ (that is the Frechét 2nd derivative of $\bar{\psi}$ with respect to $\mathbf{E}_r{}'$ calculated for $\mathbf{E}_r{}' \equiv \mathbf{E}^+$). Therefore, by dividing both sides of (3.22) by α and by introducing (3.23) we get

$$\frac{\bar{\sigma}(t)}{\alpha_2} = -\delta\mathbf{\Phi}(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{B}_r{}'\cdot\dot{\mathbf{B}}_r{}' - \frac{1}{\alpha}\mathbf{o}(\|\alpha\mathbf{B}_r{}'\|_h)\cdot\dot{\mathbf{B}}_r{}' \geq 0. \tag{3.24}$$

From definition of operator $\mathbf{o}(\|\cdot\|_h)$ it follows that the limit

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\mathbf{o}(\|\alpha\mathbf{B}_r{}'\|_h) = \boldsymbol{\phi} \tag{3.25}$$

holds true for any bounded history $\mathbf{B}_r{}'$. Since not only $\mathbf{B}_r{}'$ but also $\dot{\mathbf{B}}_r{}'$ are bounded, it is immediate to argue from (3.25) that

$$\frac{1}{\alpha}\mathbf{o}(\|\alpha\mathbf{B}_r{}'\|_h)\cdot\dot{\mathbf{B}}_r{}' = o(\alpha). \tag{3.26}$$

By means of this equation, relation (3.24) can be expressed in the form

$$\frac{\bar{\sigma}(t)}{\alpha^2} = -\delta\mathbf{\Phi}(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{B}_r{}'\cdot\dot{\mathbf{B}}_r{}' + o(\alpha).$$

That is

$$\bar{\sigma}(t) = -\alpha^2\delta\mathbf{\Phi}(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{B}_r{}'\cdot\dot{\mathbf{B}}_r{}' + o(\alpha^3) \geq 0. \tag{3.28}$$

Since α does not depend on $\mathbf{B}_r{}'$, $\dot{\mathbf{B}}_r{}'$, $\mathbf{E}^+$ and $\mathbf{E}$, it is apparent from (3.28) that one can always assume α to be so small that the relation

$$\bar{\sigma}(t) = -\alpha^2\delta\mathbf{\Phi}(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{B}_r{}'\cdot\dot{\mathbf{B}}_r{}' \geq 0 \tag{3.29}$$

is met with any desired degree of accuracy. This clearly implies that

$$-\delta\mathbf{\Phi}(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{B}_r{}'\cdot\dot{\mathbf{B}}_r{}' \geq 0. \tag{3.30}$$

Let us now consider any strain history $\mathbf{C}^t(s)$ such that

$$\mathbf{C}^t(0) = k\mathbf{H}, \tag{3.31}$$

where k is any real number and $\mathbf{H}$ any symmetric tensor. It will be assumed that both $\mathbf{C}^t(s)$ and $\dot{\mathbf{C}}^t(s)$ are bounded in the interval $0 < s < \infty$. Let $\mathbf{H}^+$ denote the constant history with value $\mathbf{H}$ and let, moreover, apply relation (3.30) to the case in which $\mathbf{B}_r{}'$ has the form

$$\mathbf{B}_r{}' = \mathbf{C}_r{}' - k\mathbf{H}^+. \tag{3.32}$$

From (3.30) and (3.32) we get

$$-\delta\mathbf{\Phi}(\mathbf{E}^+;\mathbf{E})\cdot(\mathbf{C}_r{}' - k\mathbf{H}^+)\cdot\dot{\mathbf{C}}_r{}' \geq 0, \tag{3.33}$$

because in the present case the time derivative of $\mathbf{B}_r{}'$ coincides with $\dot{\mathbf{C}}_r{}'$. By setting

$$h \overset{\text{def}}{=} -\delta\mathbf{\Phi}(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{C}_r{}'\cdot\dot{\mathbf{C}}_r{}', \tag{3.34}$$

inequality (3.33) can be expressed in the form

$$h + k\delta\mathbf{\Phi}(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{H}^+\cdot\dot{\mathbf{C}}_r{}' \geq 0. \tag{3.35}$$

From the hypotheses of derivability assumed for $\bar{\psi}$ [and hence for $\Phi(\cdot;\cdot)$], and from the hypothesis that $C_r^t(s)$ and $\dot{C}_r^t(s)$ are bounded, it follows that the scalar h defined by (3.34) is finite. This scalar, moreover, does not depend on k. Indeed the only quantity that k affects is the present value of the strain history $C^t(s)$, according to assumption (3.31). As it should be clear from definition (3.34), however, the scalar h depend on the past history of $C^t(s)$, not on its present value $C^t(0)$. Since no continuity assumptions have been introduced for the strain histories that the material can undergo, the history $C^t(s)$ need not be continuous at $s = 0$. By appropriately choosing this history, therefore, we can make the scalar h assume values that are independent of the value of k. The scalar k, however, can be chosen arbitrarily. It follows, therefore, that relation (3.35) must be met for arbitrary values of k and for any value of h given by (3.34). Consequently, the following relations

$$\delta\Phi(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{H}^+\cdot\dot{\mathbf{C}}_r^t = 0 \tag{3.36}$$

and

$$h \geq 0 \tag{3.37}$$

must necessarily hold true.

From eqn (3.17) and from the definition of Frechét derivative (see, e.g. [4], pp. 299–300), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{1}{\alpha}\bar{\sigma}_A(t)\right]_{\alpha=0} = \delta_{\mathbf{E}}{}^2\bar{\psi}(\mathbf{E}^+;\mathbf{E}|\mathbf{M}^+,\mathbf{A}(s)). \tag{3.38}$$

Here the operator $\delta_{\mathbf{E}}{}^2\bar{\psi}(\mathbf{E}^+;\mathbf{E}|\cdot,\cdot)$ denotes the Frechét 2nd derivative of $\bar{\psi}(\mathbf{E}_r^t,\mathbf{E})$ with respect to $\mathbf{E}_r^t$, calculated for $\mathbf{E}_r^t \equiv \mathbf{E}^+$. The history $\mathbf{A}(s)$ appearing in eqn (3.38) is, of course, the one already introduced in (3.12), while the history $\mathbf{M}^+$ is the history with constant value $\mathbf{M}$. In deriving eqn (3.38), moreover, use of eqns (3.14) and (3.15) has been made. By adopting the same notations introduced in (2.6) and (3.23), eqn (3.38) can be written in the more convenient form

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{1}{\alpha}\bar{\sigma}_A(t)\right]_{\alpha=0} = \delta_{\mathbf{E}}\Phi(\mathbf{E}^+;\mathbf{E}|\mathbf{A}(s))\cdot\mathbf{M}^+ = \delta\Phi(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{M}^+\cdot\mathbf{A}(s). \tag{3.39}$$

By applying eqn (3.36) to the case in which $\mathbf{H}^+ \equiv \mathbf{M}^+$ and $\dot{\mathbf{C}}_r^t \equiv \mathbf{A}(s)$, we get

$$\delta\Phi(\mathbf{E}^+;\mathbf{E})\cdot\mathbf{M}^+\cdot\mathbf{A}(s) = 0. \tag{3.40}$$

In view of this equation, eqn (3.39) yields

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{1}{\alpha}\bar{\sigma}_A(t)\right]_{\alpha=0} = 0; \tag{3.41}$$

which owing to (3.18) gives

$$\mathrm{tr}\left[\mathbf{U}^{-1}\int_0^\infty \dot{\mathbf{K}}(\mathbf{E},s)\mathbf{M}s\,\mathrm{d}s\mathbf{U}^{-1}\mathbf{M}\right] = 0. \tag{3.42}$$

Since eqn (3.42) must be satisfied for all symmetric tensors $\mathbf{M}$, it implies that

$$\int_0^\infty \dot{\mathbf{K}}(\mathbf{E},s)s\,\mathrm{d}s = 0. \tag{3.43}$$

Therefore, by remembering that $\dot{\mathbf{K}}(\mathbf{E},s) = -(\mathrm{d}/\mathrm{d}s)\mathbf{K}(\mathbf{E},s)$, and by integrating by parts the l.h.s.

of (3.43), it can be inferred that the relation

$$\lim_{x\to\infty}\left\{-[\mathbf{K}(\mathbf{E},s)s]_0^x+\int_0^\infty \mathbf{K}(\mathbf{E},s)\,\mathrm{d}s\right\}=\lim_{x\to\infty}\left\{-[\mathbf{K}(\mathbf{E},s)s]_0^x+\int_0^b \mathbf{K}(\mathbf{E},s)\,\mathrm{d}s+\int_b^x \mathbf{K}(\mathbf{E},s)\,\mathrm{d}s\right\}=0 \tag{3.44}$$

must be valid for any b in the interval $[0, x]$. If ξ_1 and ξ_2 denote two appropriate values of s, belonging respectively to the intervals $[0, b]$ and $[b, x]$, we can write eqn $(3.44)_2$ in the form

$$\lim_{x\to\infty}\{-[\mathbf{K}(\mathbf{E},s)s]_0^x+\mathbf{K}(\mathbf{E},\xi_1)b+\mathbf{K}(\mathbf{E},\xi_2)(x-b)\}=\lim_{x\to\infty}\{[\mathbf{K}(\mathbf{E},\xi_2)-\mathbf{K}(\mathbf{E},x)]x\}+\lim_{x\to\infty}\{[\mathbf{K}(\mathbf{E},\xi_1)-\mathbf{K}(\mathbf{E},\xi_2)]b\}=0. \tag{3.45}$$

Since $\mathbf{K}(\mathbf{E}, s)$ is bounded in $[0, \infty)$, the first limit in $(3.45)_2$ diverges unless

$$\lim_{x\to\infty}\mathbf{K}(\mathbf{E},\xi_2)=\lim_{x\to\infty}\mathbf{K}(\mathbf{E},x), \tag{3.46}$$

in which case the limit may tend to a finite value. Since, however, the second limit in $(3.45)_2$ tends to a finite value which depends on b, and since b can be arbitrary in $[0, x]$, it follows that eqn $(3.45)_2$ can be satisfied only if

$$\lim_{x\to\infty}\{[\mathbf{K}(\mathbf{E},\xi_2)-\mathbf{K}(\mathbf{E},x)]x\}=0 \tag{3.47}$$

and

$$\lim_{x\to\infty}\{[\mathbf{K}(\mathbf{E},\xi_1)-\mathbf{K}(\mathbf{E},\xi_2)]b\}=0. \tag{3.48}$$

In view of the arbitrariety of b and, thus, of ξ_1, eqn (3.48) implies, therefore, that

$$\mathbf{K}(\mathbf{E},s)\equiv\mathbf{K}_0(\mathbf{E}); \tag{3.49}$$

that is $\mathbf{K}$ cannot depend on s. Relation (3.49) is also sufficient to ensure that (3.47) is met. A time-derivation of (3.49) yields (3.11) and, thus, the theorem is finally proved.

In more expressive terms the above theorem states that if the standard thermodynamic theory is followed, the isothermal first order approximation of a simple material with fading memory is a perfectly elastic material. Linear viscoelastic materials, therefore, fall outside the scope of this theory. A result which is rather surprising both in view of the general character of the constitutive equations on which the theory is based and in view of the fact that linear viscoelastic materials have been shown in[1] to provide the first order approximation of the purely mechanical theory of simple materials with fading memory. Owing to this result it can reasonably be concluded that the thermodynamic approach adopted in[2, 3] may turn out to be unduly restrictive. Essentially the same conclusion has been reached on different grounds in a recent paper[5].

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